

NON-COMMUTATIVE CALLEBAUT INEQUALITY

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ABSTRACT. We present an operator version of the Callebaut inequality involving the interpolation paths and apply it to the weighted operator geometric means. We also establish a matrix version of the Callebaut inequality and as a consequence obtain an inequality including the Hadamard product of matrices.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and let I be the identity operator. In the case when $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field and denote its identity by I_n . The cone of positive operators is denoted by $\mathbb{B}(\mathcal{H})_+$. We also denote the set of all invertible positive operators (positive definite matrices, resp.) by \mathcal{P} (\mathcal{P}_n , resp.).

The axiomatic theory for operator means for positive operators acting on Hilbert space operators was established by Kubo and Ando [14]. A binary operation $\sigma : \mathbb{B}(\mathcal{H})_+ \times \mathbb{B}(\mathcal{H})_+ \rightarrow \mathbb{B}(\mathcal{H})_+$ is called an *operator mean* provided that

- (i) $I\sigma I = I$;
- (ii) $C^*(A\sigma B)C \leq (C^*AC)\sigma(C^*BC)$;
- (iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $(A_n\sigma B_n) \downarrow A\sigma B$, where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \dots$ and $A_n \rightarrow A$ as $n \rightarrow \infty$ in the strong operator topology;
- (iv)

$$A \leq B \text{ \& } C \leq D \implies A\sigma C \leq B\sigma D. \quad (1.1)$$

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It follows from the general theory of means [6, Theorem 5.7]) that

$$\sum_{j=1}^m (A_j \sigma B_j) \leq \left(\sum_{j=1}^m A_j \right) \sigma \left(\sum_{j=1}^m B_j \right). \quad (1.2)$$

There exists an affine order isomorphism between the class of operator means and the class of positive monotone operator functions f defined on $[0, \infty)$ with $f(1) = 1$ via $f(t)I = I\sigma(tI)$ ($t \geq 0$). In addition, $A\sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$. The operator monotone function f is called the *representing function* of σ . Using a limit argument by $A_\varepsilon = A + \varepsilon I$, one can extend the definition of $A\sigma B$ to positive operators (positive semidefinite matrices) as well. The dual σ^\perp of an operator mean σ with the representing function f is the operator mean with representing function $t/f(t)$. So that $A\sigma^\perp B = A^{1/2}(A^{-1/2}BA^{-1/2})f(A^{-1/2}BA^{-1/2})^{-1}A^{1/2}$. The operator means corresponding to the positive operator monotone functions $f_\sharp(t) = t^{1/2}$, $f_{\sharp p}(t) = t^p$ ($0 \leq p \leq 1$) are the operator geometric mean $A\sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}} A^{\frac{1}{2}}$ and the operator weighted geometric mean $A\sharp_p B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^p A^{\frac{1}{2}}$.

For an operator mean σ satisfying $A\sigma B = B\sigma A$ a parametrized operator mean σ_t is called an *interpolational path* for σ (see [6, Section 5.3] and [8]) if it satisfies

- (1) $A\sigma_0 B = A$, $A\sigma_{1/2} B = A\sigma B$ and $A\sigma_1 B = B$;
- (2) $(A\sigma_p B)\sigma(A\sigma_q B) = A\sigma_{(p+q)/2} B$, for all $p, q \in [0, 1]$;
- (3) the map $t \mapsto A\sigma_t B$ is norm continuous for each A, B .

It is easy to see that the set of all $r \in [0, 1]$ satisfying

$$(A\sigma_p B)\sigma_r(A\sigma_q B) = A\sigma_{rp+(1-r)q} B. \quad (1.3)$$

for all p, q is a convex subset of $[0, 1]$ including 0 and 1. Therefore (1.3) is valid for all $p, q, r \in [0, 1]$, cf. [8, Lemma 1].

The power means $Am_r B = A^{1/2} \left((1 + (A^{-1/2}BA^{-1/2})^r) / 2 \right)^{1/r} A^{1/2}$ ($r \in [-1, -1]$) are typical examples of interpolational means, whose interpolational paths, by [6, Theorem 5.24], are

$$Am_{r,t} B = A^{\frac{1}{2}} \left(1 - t + t \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^r \right)^{\frac{1}{r}} A^{\frac{1}{2}} \quad (t \in [0, 1]). \quad (1.4)$$

One of the fundamental inequalities in mathematics is the Cauchy–Schwarz inequality. There are many generalizations and applications of this inequality; see the monograph [4]. There are some Cauchy–Schwarz type inequalities for Hilbert space operators and matrices involving

unitarily invariant norms given by Jocić [11] and Kittaneh [13]. Moreover, Niculescu [17], Joița [12], Ilišević–Varošanec [10], Moslehian–Persson [16], Arambasić–Bakić–Moslehian [2] have investigated the Cauchy–Schwarz inequality and its various reverses in the framework of C^* -algebras and Hilbert C^* -modules. An application of the covariance-variance inequality to the Cauchy–Schwarz inequality was obtained by Fujii–Izumino–Nakamoto–Seo [9]. Some operator versions of the Cauchy–Schwarz inequality with simple conditions for the case of equality are presented by Fujii [7].

In 1965, Callebaut [3] gave the following refinement of the Cauchy–Schwarz inequality:

Given a real number s , non-proportional sequences of positive real numbers $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$, the function $f(r, s) = (\sum_{i=1}^n a_i^{s+r} b_i^{s-r}) (\sum_{i=1}^n a_i^{s-r} b_i^{s+r})$ is increasing in $0 \leq |r| \leq 1$. If $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ are proportional, then this expression is independent of r .

Thus one can obtain many well-ordered inequalities lying between the left and the right sides of the Cauchy–Schwarz inequality. In particular, if $0 \leq t \leq s \leq \frac{1}{2}$ or $\frac{1}{2} \leq s \leq t \leq 1$, then

$$\begin{aligned} \left(\sum_{j=1}^m a_j^{1/2} b_j^{1/2} \right)^2 &\leq \left(\sum_{j=1}^m a_j^s b_j^{1-s} \right) \left(\sum_{j=1}^m a_j^{1-s} b_j^s \right) \leq \left(\sum_{j=1}^m a_j^t b_j^{1-t} \right) \left(\sum_{j=1}^m a_j^{1-t} b_j^t \right) \\ &\leq \left(\sum_{j=1}^m a_j \right) \left(\sum_{j=1}^m b_j \right) \end{aligned} \quad (1.5)$$

for all positive real numbers a_j, b_j ($1 \leq j \leq m$). This triple inequality is well-known as the Callebaut inequality. Applying Hölder’s inequality, McLaughlin and Metcalf [15] obtained it in a simple fashion. The method of Callebaut may be used for finding other interesting refinements of classical inequalities, see [5, 1] and references therein. Wada [18] gave an operator version of the Callebaut inequality by showing that if A and B are positive operators on a Hilbert space and if σ is an operator mean, then

$$(A \sharp B) \otimes (A \sharp B) \leq \frac{1}{2} \{ (A\sigma B) \otimes (A\sigma^\perp B) + (A\sigma^\perp B) \otimes (A\sigma B) \} \leq \frac{1}{2} \{ (A \otimes B) + (B \otimes A) \}.$$

The purpose of the paper is to present some noncommutative versions of the Callebaut inequality. More precisely, we give an operator Callebaut inequality involving the interpolation paths and apply it to the weighted operator geometric means. We also establish a matrix version of the Callebaut inequality and as a consequence obtain an inequality including the Hadamard product of matrices.

2. CALLEBAUT INEQUALITY FOR HILBERT SPACE OPERATORS

Our first operator version of (1.5) reads as follows:

Theorem 2.1. *Let $A_j, B_j \in \mathcal{P}$ ($1 \leq j \leq m$) and σ be an operator mean. Then*

$$\sum_{j=1}^m (A_j \sharp B_j) \leq \left(\sum_{j=1}^m A_j \sigma B_j \right) \sharp \left(\sum_{j=1}^m A_j \sigma^\perp B_j \right) \leq \left(\sum_{j=1}^m A_j \right) \sharp \left(\sum_{j=1}^m B_j \right). \quad (2.1)$$

Furthermore, if σ_t is an interpolational path for σ such that $\sigma_t^\perp = \sigma_{1-t}$, then

$$\left(\sum_{j=1}^m A_j \sigma_s B_j \right) \sharp \left(\sum_{j=1}^m A_j \sigma_{1-s} B_j \right) \leq \left(\sum_{j=1}^m A_j \sigma_t B_j \right) \sharp \left(\sum_{j=1}^m A_j \sigma_{1-t} B_j \right) \quad (2.2)$$

for s between t and $1-t$.

Proof. Let f be the representing function of σ . Then

$$\begin{aligned} (A\sigma B) \sharp (A\sigma^\perp B) &= (A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}) \sharp (A^{1/2} (A^{-1/2} B A^{-1/2}) f(A^{-1/2} B A^{-1/2})^{-1} A^{1/2}) \\ &= A^{1/2} (f(A^{-1/2} B A^{-1/2}) \sharp ((A^{-1/2} B A^{-1/2}) f(A^{-1/2} B A^{-1/2})^{-1})) A^{1/2} \\ &= A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2} \\ &= A \sharp B \end{aligned} \quad (2.3)$$

for all $A, B \in \mathcal{P}$. It follows from (1.2) that

$$\begin{aligned} \sum_{j=1}^m (A_j \sigma B_j) &\leq \left(\sum_{j=1}^m A_j \right) \sigma \left(\sum_{j=1}^m B_j \right) \\ \sum_{j=1}^m (A_j \sigma^\perp B_j) &\leq \left(\sum_{j=1}^m A_j \right) \sigma^\perp \left(\sum_{j=1}^m B_j \right), \end{aligned}$$

whence

$$\begin{aligned} \left(\sum_{j=1}^m A_j \sigma B_j \right) \sharp \left(\sum_{j=1}^m A_j \sigma^\perp B_j \right) &\leq \left(\left(\sum_{j=1}^m A_j \right) \sigma \left(\sum_{j=1}^m B_j \right) \right) \sharp \left(\left(\sum_{j=1}^m A_j \right) \sigma^\perp \left(\sum_{j=1}^m B_j \right) \right) \\ &\quad \text{(by (1.1))} \\ &= \left(\sum_{j=1}^m A_j \right) \sharp \left(\sum_{j=1}^m B_j \right), \quad \text{(by (2.3))} \end{aligned}$$

which gives the second inequality of (2.1). Now we prove the first inequality of (2.1):

$$\begin{aligned} \left(\sum_{j=1}^m A_j \sigma B_j \right) \sharp \left(\sum_{j=1}^m A_j \sigma^\perp B_j \right) &\geq \sum_{j=1}^m ((A_j \sigma B_j) \sharp (A_j \sigma^\perp B_j)) \\ &= \sum_{j=1}^m A_j \sharp B_j. \end{aligned} \quad (\text{by (2.3)})$$

Next we prove (2.2). Replacing A_j and B_j by $A_j \sigma_t B_j$ and $A_j \sigma_{1-t} B_j$, respectively, in (2.1) and noting to $\sigma_s^\perp = \sigma_{1-s}$ we get

$$\begin{aligned} \left(\sum_{j=1}^m (A_j \sigma_t B_j) \sigma_s (A_j \sigma_{1-t} B_j) \right) \sharp \left(\sum_{j=1}^m (A_j \sigma_t B_j) \sigma_{1-s} (A_j \sigma_{1-t} B_j) \right) \\ \leq \sum_{j=1}^m (A_j \sigma_t B_j) \sharp \sum_{j=1}^m (A_j \sigma_{1-t} B_j). \end{aligned}$$

The first term of the inequality above, by (1.3), is

$$\begin{aligned} \left(\sum_{j=1}^m (A_j \sigma_t B_j) \sigma_s (A_j \sigma_{1-t} B_j) \right) \sharp \left(\sum_{j=1}^m (A_j \sigma_t B_j) \sigma_{1-s} (A_j \sigma_{1-t} B_j) \right) \\ = \left(\sum_{j=1}^m (A_j \sigma_{ts+(1-t)(1-s)} B_j) \right) \sharp \left(\sum_{j=1}^m (A_j \sigma_{(1-t)s+t(1-s)} B_j) \right) \\ = \left(\sum_{j=1}^m (A_j \sigma_{ts+(1-t)(1-s)} B_j) \right) \sharp \left(\sum_{j=1}^m (A_j \sigma_{1-(ts+(1-t)(1-s))} B_j) \right). \end{aligned}$$

If s is between t and $1-t$, then there is $s_0 \in [0, 1]$ such that $s = ts_0 + (1-t)(1-s_0)$. This shows that the desired inequality holds. \square

It follows from (1.4) that $Am_{0,t}B = A \sharp_t B$. Due to $A \sharp_t^\perp B = A \sharp_{1-t} B$, we infer that

Corollary 2.2. *Let $A_j, B_j \in \mathcal{P}$. Then*

$$\sum_{j=1}^m A_j \sharp B_j \leq \left(\sum_{j=1}^m A_j \sharp_s B_j \right) \sharp \left(\sum_{j=1}^m A_j \sharp_{1-s} B_j \right) \leq \sum_{j=1}^m A_j \sharp \sum_{j=1}^m B_j.$$

Moreover,

$$\left(\sum_{j=1}^m A_j \sharp_s B_j \right) \sharp \left(\sum_{j=1}^m A_j \sharp_{1-s} B_j \right) \leq \left(\sum_{j=1}^m A_j \sharp_t B_j \right) \sharp \left(\sum_{j=1}^m A_j \sharp_{1-t} B_j \right)$$

for s between t and $1-t$.

Taking positive scalars a_j and b_j for A_j and B_j , respectively, in Theorem 2.1 we get

Corollary 2.3. *The classical Callebaut inequality (1.5) holds.*

3. CALLEBAUT INEQUALITY FOR MATRICES

To achieve some Callebaut inequalities for matrices, we need the following lemma.

Lemma 3.1. *Let $0 \leq r \leq 1$. Then $A^r + A^{-r} \leq A + A^{-1}$ for all $A \in \mathcal{P}_n$.*

Proof. Suppose that $A = U\Gamma U^*$ with unitary U and diagonal matrix Γ . Then

$$\begin{aligned} A^r + A^{-r} &= U(\Gamma^r + \Gamma^{-r})U^* \\ &\leq U(\Gamma + \Gamma^{-1})U^* = A + A^{-1} \end{aligned}$$

since $t^r + t^{-r} \leq t + t^{-1}$ for any positive real number t and $0 \leq r \leq 1$. □

Theorem 3.2. *The function*

$$f(t) = A^{1+t} \otimes B^{1-t} + A^{1-t} \otimes B^{1+t}$$

is decreasing on the interval $[-1, 0]$, increasing on the interval $[0, 1]$ and attains its minimum at $t = 0$ for all $A, B \in \mathcal{P}_n$.

Proof. Let $0 \leq \alpha \leq \beta$. Taking $0 \leq r = \frac{\alpha}{\beta} \leq 1$ and replacing A by $A^\beta \otimes B^{-\beta}$ in Lemma 3.1, we get

$$A^\alpha \otimes B^{-\alpha} + A^{-\alpha} \otimes B^\alpha \leq A^\beta \otimes B^{-\beta} + A^{-\beta} \otimes B^\beta.$$

This further implies that

$$\begin{aligned} (A^{1/2} \otimes B^{1/2}) (A^\alpha \otimes B^{-\alpha} + A^{-\alpha} \otimes B^\alpha) (A^{1/2} \otimes B^{1/2}) \\ \leq (A^{1/2} \otimes B^{1/2}) (A^\beta \otimes B^{-\beta} + A^{-\beta} \otimes B^\beta) (A^{1/2} \otimes B^{1/2}), \end{aligned}$$

whence

$$A^{1+\alpha} \otimes B^{1-\alpha} + A^{1-\alpha} \otimes B^{1+\alpha} \leq A^{1+\beta} \otimes B^{1-\beta} + A^{1-\beta} \otimes B^{1+\beta}$$

Note that $f(t) = f(-t)$, so f is decreasing on the interval $[-1, 0]$. The last statement on minimum point is obvious from the preceding ones. □

Corollary 3.3. *The function*

$$g(t) = A^t \otimes B^{1-t} + A^{1-t} \otimes B^t$$

is decreasing on $[0, 1/2]$, increasing on $[1/2, 1]$ and attains its minimum at $t = \frac{1}{2}$ for all $A, B \in \mathcal{P}_n$.

Proof. The proof follows by replacing A, B by $A^{1/2}, B^{1/2}$ in Theorem 3.2, respectively, and then replacing $\frac{1+t}{2}$ by t . \square

The following theorem is our second version of the Callebaut inequality (1.5).

Theorem 3.4. *Let $A_j, B_j \in \mathcal{P}_n$, $1 \leq j \leq m$. Then*

$$\begin{aligned} 2 \sum_{j=1}^m (A_j \sharp B_j) \otimes \sum_{j=1}^m (A_j \sharp B_j) \\ \leq \sum_{j=1}^m (A_j \sharp_s B_j) \otimes \sum_{j=1}^m (A_j \sharp_{1-s} B_j) + \sum_{j=1}^m (A_j \sharp_{1-s} B_j) \otimes \sum_{j=1}^m (A_j \sharp_s B_j) \\ \leq \sum_{j=1}^m (A_j \sharp_t B_j) \otimes \sum_{j=1}^m (A_j \sharp_{1-t} B_j) + \sum_{j=1}^m (A_j \sharp_{1-t} B_j) \otimes \sum_{j=1}^m (A_j \sharp_t B_j) \\ \leq \sum_{j=1}^m A_j \otimes \sum_{j=1}^m B_j + \sum_{j=1}^m B_j \otimes \sum_{j=1}^m A_j. \end{aligned}$$

for $0 \leq t \leq s \leq \frac{1}{2}$ or $\frac{1}{2} \leq s \leq t \leq 1$.

Proof. In order to prove the above inequalities we first prove that the function

$$f(t) = \sum_{j=1}^m (A_j \sharp_t B_j) \otimes \sum_{j=1}^m (A_j \sharp_{1-t} B_j) + \sum_{j=1}^m (A_j \sharp_{1-t} B_j) \otimes \sum_{j=1}^m (A_j \sharp_t B_j)$$

is decreasing on $[0, 1/2]$, increasing on $[1/2, 1]$ and attains its minimum at $t = \frac{1}{2}$. By Corollary 3.3 the function $t \rightarrow C_i^t \otimes C_j^{1-t} + C_i^{1-t} \otimes C_j^t$, where $C_j = A_j^{-1/2} B_j A_j^{-1/2}$, is decreasing on $[0, 1/2]$, increasing on $[1/2, 1]$ and attains its minimum at $t = \frac{1}{2}$. Hence the function

$$\begin{aligned} t \rightarrow \left(A_i^{1/2} \otimes A_j^{1/2} \right) \left(C_i^t \otimes C_j^{1-t} + C_i^{1-t} \otimes C_j^t \right) \left(A_i^{1/2} \otimes A_j^{1/2} \right) \\ = (A_i \sharp_t B_i) \otimes (A_j \sharp_{1-t} B_j) + (A_i \sharp_{1-t} B_i) \otimes (A_j \sharp_t B_j) \end{aligned}$$

is decreasing on $[0, 1/2]$, increasing on $[1/2, 1]$ and attains its minimum at $t = \frac{1}{2}$. Since

$$\begin{aligned} & \sum_{j=1}^m (A_j \sharp_t B_j) \otimes \sum_{j=1}^m (A_j \sharp_{1-t} B_j) + \sum_{j=1}^m (A_j \sharp_{1-t} B_j) \otimes \sum_{j=1}^m (A_j \sharp_t B_j) \\ &= \sum_{i, j=1}^m \left((A_i \sharp_t B_i) \otimes (A_j \sharp_{1-t} B_j) + (A_i \sharp_{1-t} B_i) \otimes (A_j \sharp_t B_j) \right) \end{aligned}$$

therefore the function $f(t)$ is decreasing on $[0, 1/2]$, increasing on $[1/2, 1]$ and attains its minimum at $t = \frac{1}{2}$. On using the fact that $f(1/2) \leq f(s) \leq f(t) \leq f(1)$, for $\frac{1}{2} \leq s \leq t \leq 1$, or $f(1/2) \leq f(s) \leq f(t) \leq f(0)$, for $\frac{1}{2} \geq s \geq t \geq 0$, we get the desired result. \square

We have the following corollary as a consequence to Theorem 3.4.

Corollary 3.5. *Let $A_j, B_j \in \mathcal{P}_n$, $1 \leq j \leq m$. Then*

$$\begin{aligned} \sum_{j=1}^m (A_j \sharp B_j) \circ \sum_{j=1}^m (A_j \sharp B_j) &\leq \sum_{j=1}^m (A_j \sharp_s B_j) \circ \sum_{j=1}^m (A_j \sharp_{1-s} B_j) \\ &\leq \sum_{j=1}^m (A_j \sharp_t B_j) \circ \sum_{j=1}^m (A_j \sharp_{1-t} B_j) \\ &\leq \sum_{j=1}^m A_j \circ \sum_{j=1}^m B_j \end{aligned}$$

for $0 \leq t \leq s \leq \frac{1}{2}$ or $\frac{1}{2} \leq s \leq t \leq 1$.

The next result is a consequence of Theorem 3.4 with $B_j = I_n$ ($1 \leq j \leq n$).

Corollary 3.6. *Let $A_j \in \mathcal{P}_n$, $1 \leq j \leq m$. Then*

$$\left(\frac{1}{m} \sum_{j=1}^m A_j^{1/2} \right) \circ \left(\frac{1}{m} \sum_{j=1}^m A_j^{1/2} \right) \leq \left(\frac{1}{m} \sum_{j=1}^m A_j^t \right) \circ \left(\frac{1}{m} \sum_{j=1}^m A_j^{1-t} \right) \leq \frac{1}{m} \sum_{j=1}^m (A_j \circ I_n) \quad (3.1)$$

for all $0 \leq t \leq 1$.

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